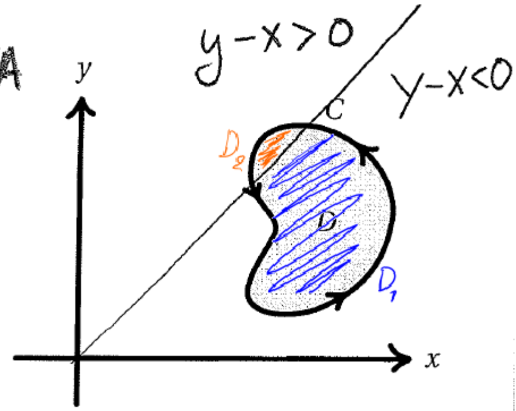
Question 1

Consider the region D in the plane bounded by the curve C as shown at right. For each part, circle the best answer. **(1 point each)**



(a) For $\mathbf{F}(x,y) = \langle \overset{P}{x+1}, \overset{Q}{y^2} \rangle$, the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is

negative	zero	positive

$= \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$

$= \iint_R 0\, dA = 0$

(b) The integral $\int_C (\overset{P}{-y}\, dx + \overset{Q}{2}\, dy)$ is

negative	zero	positive

$= \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \iint_R 1\, dA$

$= \text{Area}(R)$

(c) The integral $\iint_D (y-x)\, dA$ is

negative	zero	positive

Split D into D_1 and D_2.

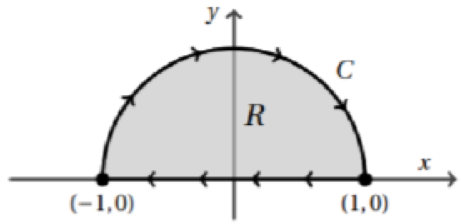
Then $I = \iint_D (y-x)\, dA = \underbrace{\iint_{D_1} (y-x)\, dA}_{<0} + \underbrace{\iint_{D_2} (y-x)\, dA}_{>0}$ and $\left|\iint_{D_1}\right| > \left|\iint_{D_2}\right|$,

so I has to be negative.

Question 2

Let $\mathbf{F}(x, y) = \left\langle x-1,\ \cos y + 2x - e^{y^2} \right\rangle$. Let R denote the solid semi-disk shown below right. Let C denote the boundary of the region R.

(a) Use Green's Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C has the orientation shown. **(3 points)**



SOLUTION:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = -\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = -\iint_R 2\, dA = -2\text{Area}(R) = -\pi.$$

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = -\pi}$$

(b) Let D denote the part of the curve C above consisting *only* of the semicircle (*not* the line segment) with the orientation shown. Compute $\int_D \mathbf{F} \cdot d\mathbf{r}$. **(3 points)**

SOLUTION:

Let C'' denote the oriented segment from $(-1, 0)$ to $(1, 0)$. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r} - \int_{C''} \mathbf{F} \cdot d\mathbf{r}$$

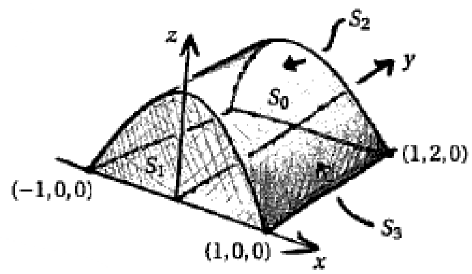
Using Part (a) we get

$$\int_{C'} \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r} + \int_{C''} \mathbf{F} \cdot d\mathbf{r} = -\pi + \int_{C''} \mathbf{F} \cdot d\mathbf{r} = -\pi + \int_{C''} P dx + Q dy = -\pi + \int_{-1}^{1} (x-1) dx = -\pi - 2.$$

$$\boxed{\int_{C'} \mathbf{F} \cdot d\mathbf{r} = -\pi - 2}$$

Question 3

. Let E be the solid region shown below, where ∂E is decomposed into the four subsurfaces S_i indicated; here the top S_0 is where $z + x^2 = 1$, the front S_1 is in the xz-plane, the back is S_2, and the bottom is S_3.



(b) Give a parameterization of S_0 and use it to directly compute the flux of $\mathbf{F} = \langle 1,\ 0,\ z+2\rangle$ through S_0 with respect to the upwards normals. **(5 points)**

$$\vec{r}(u,v) = \langle u, v, 1-u^2\rangle \qquad D = \{-1 \le u \le 1 \text{ and } 0 \le v \le 2\}$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & -2u \\ 0 & 1 & 0 \end{vmatrix} = \langle 2u, 0, 1\rangle$$

$$\text{Flux} = \int_0^2\int_{-1}^1 \underbrace{\langle 1, 0, 3-u^2\rangle}_{\vec{F}(\vec{r}(u,v))} \cdot \overbrace{\langle 2u, 0, 1\rangle\, du\,dv}^{\vec{r}_u\times\vec{r}_v\, du\,dv}$$

$$= \int_0^2\int_{-1}^1 2u + 3 - u^2\, du\,dv = \int_0^2 u^2 + 3u - \tfrac{1}{3}u^3 \Big|_{u=-1}^{u=1} dv = 2 \cdot 16/3$$

$$= \int_0^2 6 - 2/3\, dv$$

$$\boxed{\iint_{S_0} \mathbf{F}\cdot d\mathbf{S} = 32/3}$$

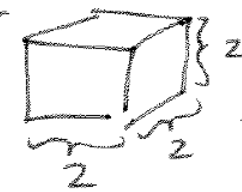
(c) The flux of $\mathbf{F}$ through exactly two of S_1, S_2, and S_3 is zero. Circle the one where the flux is **nonzero**: **(1 point)** $\boxed{S_1 \quad S_2 \quad S_3}$

Since $\vec{F}$ has no y-component & the normal $\vec{n}$ to S_1 and S_2 points along y axis, the flux through S_1 and S_2 is zero.

Question 4

Let S be the surface in $\mathbb{R}^3$ which is the boundary of the solid cube $D = \{-1 \le x \le 1, -1 \le y \le 1, -1 \le z \le 1\}$. For $\mathrm{F}(x, y, z) = \langle yz^2 + e^z + x,\ ze^z + x + y,\ xe^x + xy + z\rangle$, compute $\iint_S \mathbf{F}\cdot\mathbf{n}\, dS$ by any valid method, where $\mathbf{n}$ is the outward-pointing unit normal vector field. **(4 points)**

$$= \iiint_{\text{Cube}} \operatorname{div} \vec{F}\, dV = \iiint_{\text{Cube}} 3\, dV = 3\,\mathrm{Vol}\Big(\Big)$$



$$= 3 \cdot 2^3 = 24$$

$$\boxed{\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = 24}$$

Question 5

. Consider the vector field $\mathbf{F} = \langle -y,\ x+z,\ x^2+z\rangle$ on $\mathbb{R}^3$.

(a) Circle the curl of $\mathbf{F}$: **(2 points)**

$$\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x+z & x^2+z \end{vmatrix}$$

$$= \langle -1, -2x, 2\rangle$$

curl $\mathbf{F}$ =	$\langle z, -y, x\rangle$	$\langle -1, 2x, 2\rangle$	$\langle 0, 1, 2x\rangle$	$\langle -1, -2x, 2\rangle$	$\langle -y, 2x, 2z\rangle$

(b) Suppose C is a closed curve in the plane P given by $x - z = 1$. Assuming C bounds a region R of area 10 in P, determine the absolute value of $\int_C \mathbf{F}\cdot d\mathbf{r}$. **(4 points)**

$$\int_C \vec{F}\cdot d\vec{r} = \iint_R (\operatorname{curl} \vec{F}) \cdot \vec{n}\, dA = \iint_R \langle -1, -2x, 2\rangle \cdot \frac{1}{\sqrt{2}}\langle 1, 0, -1\rangle\, dA$$

Stokes' Theorem

$$\iint_R \frac{1}{\sqrt{2}}(-1-2)\, dA = -\frac{3}{\sqrt{2}}\,\mathrm{Area}(R) = \frac{-30}{\sqrt{2}}$$

Normal to R comes from eqn for plane: $x - z = 1$

$$\rightsquigarrow \langle 1, 0, -1\rangle \underset{\text{make unit}}{\rightsquigarrow} \vec{n} = \frac{1}{\sqrt{2}}\langle 1, 0, -1\rangle$$

$$\boxed{\left|\int_C \mathbf{F}\cdot d\mathbf{r}\right| = 30/\sqrt{2}}$$

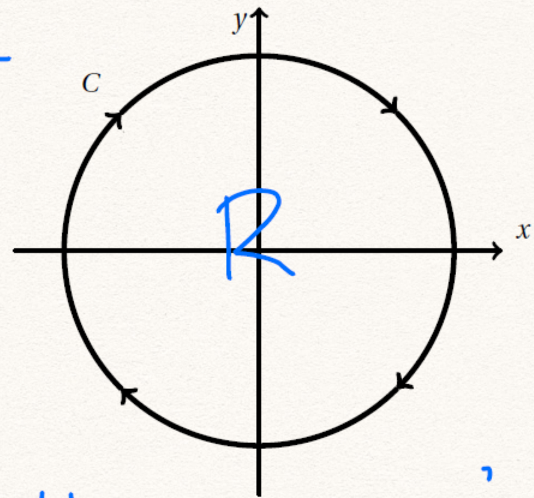
Question 6

Let C be the circle of radius 1 centered at the origin and oriented as shown below right. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = \left(\overset{P}{e^x y^2 - x^2 y}, \overset{Q}{2e^x y + x y^2}\right)$ by any valid method. **(3 points)**

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2e^x y + y^2 - 2e^x y + x^2 = x^2 + y^2$$

$$\int_C \vec{F}\cdot d\vec{r} = -\iint_R (x^2+y^2)\,dA = -\int_0^1 \int_0^{2\pi} r^2 \cdot r\, d\theta\, dr$$

$$= -\int_0^1 2\pi r^3\, dr = -\frac{\pi}{2}$$

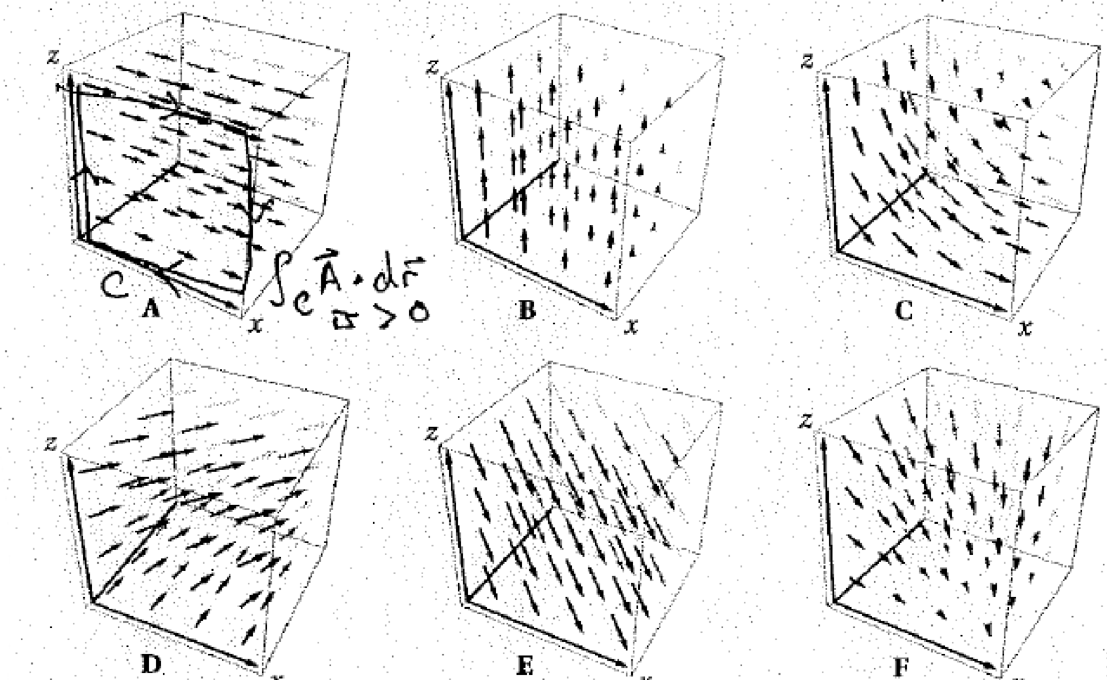


oriented clockwise

$\int_C \mathbf{F}\cdot d\mathbf{r} = -\pi/2$

Question 7

Here are plots of six vector fields on the box where $0 \le x \le 1$, $0 \le y \le 1$, and $0 \le z \le 1$. For each part, circle the best answer. **(1 point each)**



The vectors should be parallel to the xy-plane. The x component should increase as z value increases, and the y component should be 1 for each vector.

(a) The the vector field given by $\langle z, 1, 0\rangle$ is: A B C **D** E F

(b) Exactly one of these vector fields has nonzero divergence. It is: A B C D E **F**

For this example, the divergence is generally: **negative** positive

The flow is sinking into the lower right corner.

(c) The vector field **A** is conservative: true **false**

(d) Exactly one of the vector fields is constant, that is, independent of position. It is: A B C D **E** F

The vectors should have the exact same length and direction

(e) The vector field curl**C** is constant. The value of curl**C** is:

$\mathbf{i}$ $\quad -\mathbf{i}$ $\quad \mathbf{j}$ $\quad$ **$-\mathbf{j}$** $\quad \mathbf{k}$ $\quad -\mathbf{k}$ $\quad \mathbf{0}$

Right hand rule