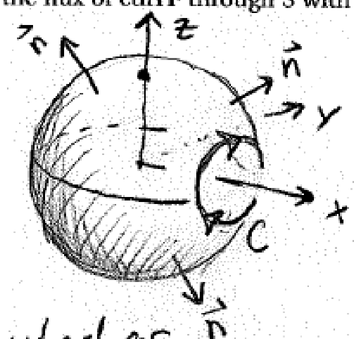
Question 1

1. (a) Consider the vector field $\mathbf{F} = \langle yz, -xz, yx \rangle$ on $\mathbb{R}^3$. Compute the curl of $\mathbf{F}$. **(2 points)**

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & -xz & yx \end{vmatrix} = (x+x)\vec{i} - (y-y)\vec{j} + (-z-z)\vec{k}$$

$$\boxed{\operatorname{curl}\mathbf{F} = \langle 2x , 0 , -2z \rangle}$$

(b) Let S be the portion of the sphere $x^2 + y^2 + z^2 = 13$ where $x \le 3$. Find the flux of $\operatorname{curl}\mathbf{F}$ through S with respect to the outward pointing unit normal vector field. **(? points)**



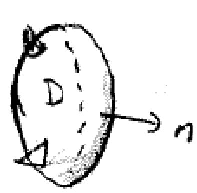
By Stokes, $\iint_S (\operatorname{curl}\vec{F})\cdot n \, dA = \int_C \vec{F}\cdot d\vec{r}$. Here C is the circle in the $x=3$ plane of radius 2 since $3^2 + y^2 + z^2 = 13 \Rightarrow y^2+z^2 = 4$, oriented as shown. So can use $\vec{r}(t) = \langle 3, 2\sin t, 2\cos t\rangle$ for $0 \le t \le 2\pi$ to param. Now $\int_C \vec{F}\cdot d\vec{r} =$

$$\int_0^{2\pi} \underbrace{\vec{F}(\vec{r}(t))}_{\langle 4\sin t\cos t,\, -6\cos t,\, 6\sin t\rangle} \cdot \underbrace{\vec{r}\,'(t)}_{\langle 0,\, 2\cos t,\, -2\sin t\rangle} \, dt = \int_0^{2\pi} -12\cos^2 t - 12\sin^2 t \, dt = \int_0^{2\pi} -12 \, dt = -24\pi$$

$$\boxed{\text{flux} = -24\pi}$$

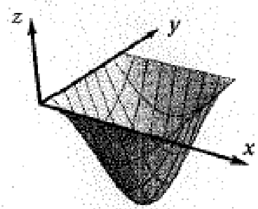
(c) Let D be the of portion of sphere $x^2 + y^2 + z^2 = 13$ where $x \ge 3$. Find the flux of $\operatorname{curl}\mathbf{F}$ through D with respect to the outward pointing unit normal vector field. **(2 points)**

For D, get the other orient on C, so ans. if − what we had in (a)



$$\boxed{\text{flux} = +24\pi}$$

Question 2



(a) Let S be the portion of the surface $z = -\sin(x)\sin(y)$ where $0 \leq x \leq \pi$ and $0 \leq y \leq \pi$ which is shown at right. Use a parameterization to find the flux of $\mathbf{F} = \langle 0, 0, 2z+1 \rangle$ through S with respect to the downward normals. **(? points)**

$$\vec{r}(u,v) = \langle u, v, -\sin u \sin v \rangle \quad 0 \leq u, v \leq \pi.$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & -\cos u \sin v \\ 0 & 1 & -\sin u \cos v \end{vmatrix} = \underbrace{\langle \cos u \sin v, \sin u \cos v, 1 \rangle}_{\text{upwards, so use } - \text{ this}}$$

$$\iint_S \vec{F} \cdot \vec{n} \, dA = \int_0^\pi \int_0^\pi \langle 0, 0, -2\sin u \sin v + 1 \rangle \cdot \underbrace{\langle *, *, -1 \rangle}_{-\vec{r}_u \times \vec{r}_v} \, du\, dv$$

$$= \int_0^\pi \int_0^\pi 2\sin u \sin v - 1 \, du\, dv$$

$$= 2\underbrace{\int_0^\pi \sin u \, du}_{= -\cos u \Big|_{u=0}^{u=\pi} = 2} \int_0^\pi \sin v \, dv - \int_0^\pi \int_0^\pi 1 \, du\, dv = 2 \cdot 2 \cdot 2 - \pi^2$$

flux = $8 - \pi^2$

(b) Let E be the region below the xy-plane and above S. Use an integral theorem to compute the flux of $\mathbf{F}$ through ∂E with respect to the outward normals. **(? points)**

Divergence Thm: $\displaystyle\iint_{\partial E} \vec{F} \cdot \vec{n} \, dA = \iiint_E \underbrace{\operatorname{div} \vec{F}}_{=2} \, dV$

$$= \int_0^\pi \int_0^\pi \int_{-\sin x \sin y}^0 2 \, dz\, dy\, dx = \int_0^\pi \int_0^\pi 2\sin x \sin y \, dy\, dx$$

$$= 2\int_0^\pi \sin x \, dx \int_0^\pi \sin y \, dy = 8$$

flux = 8

(c) Your answers in (b) and (c) should differ. Explain what accounts for the difference. **(1 point)**

∂E consists of both S and the square $R = \{0 \leq x, y \leq \pi\}$ in the xy plane. The flux through R is

$$\iint_R \vec{F} \cdot \vec{n} \, dA = \iint_R \langle 0,0,1 \rangle \cdot \langle 0,0,1 \rangle \, dA = \text{Area}(R) = \pi^2$$

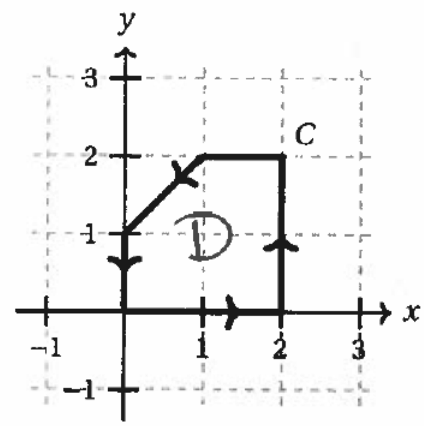
Question 3

Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \langle \underbrace{y + 2\cos(x)}_{P}, \underbrace{3x + e^{y^2}}_{Q} \rangle$ and C is the oriented curve shown. **(5 points)**

$$\int_C \vec{F} \cdot dr = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$

$$= \iint_D 3 - 1 \, dA = 2\,\text{Area}(D)$$

$$= 2\,(7/2) = 7$$



$\int_C \mathbf{F} \cdot d\mathbf{r} = \boxed{+7}$

Question 4

For this problem, $\mathbf{G} = \langle yz + 2x^2,\ 2xy,\ xy^2 \rangle$ and S is the boundary of the cube $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le$ oriented with the outward pointing normal vector $\mathbf{n}$. Circle the best response for each of the following.

(a) $\iint_S \mathbf{G} \cdot \mathbf{n}\, dS =$ | −5 | −3 | −1 | 0 | 1 | 3 | 5 | **(2 points)**

(b) $\iint_S (\text{curl}\,\mathbf{G}) \cdot \mathbf{n}\, dS$ is | negative | zero | positive | **(1 point)**

Scratch Space

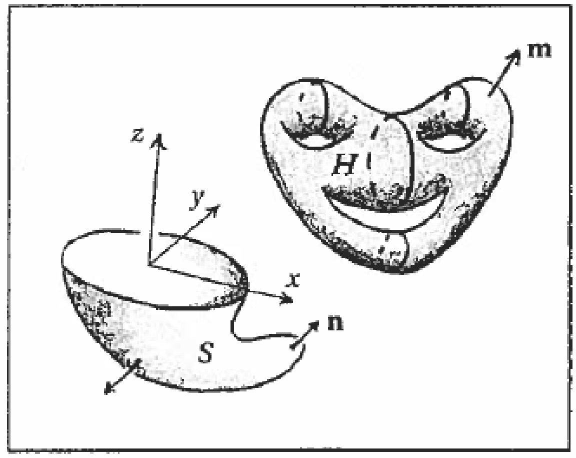
a) $\text{Flux} = \iiint_{\text{Cube}} \text{div}\,\vec{G}\, dV = \int_0^1 \int_0^1 \int_0^1 4x + 2x + 0 \, dx\, dy\, dz$

$$= \int_0^1 6x\, dx = 3x^2 \Big|_0^1 = 3$$

b) Since S is closed (no boundary), $\iint_S (\text{curl}\,\vec{G}) \cdot \vec{n}\, dS = 0$.

Question 5

Let S and H be the surfaces at right; the boundary of S is the unit circle in the xy-plane, and H has no boundary. **1 pt each**



(a) The integral $\iint_H x^2y^2 + z^2\,dS$ is:

negative	zero	positive

Since integrand is pos. on H

(b) The vector field $\mathbf{F} = \langle y+z,\ -x,\ yz\rangle$ has $\operatorname{curl}\mathbf{F} = \langle z,\ 1,\ -2\rangle$.

The flux $\iint_S (\operatorname{curl}\mathbf{F}) \cdot \mathbf{n}\, dS$ is:

-5π	-4π	-3π	-2π	$-\pi$	0	π	2π	3π	4π	5π

(c) For $\mathbf{G} = \langle x,\ y,\ z\rangle$, the flux $\iint_S \mathbf{G} \cdot \mathbf{n}\, dS$ is:

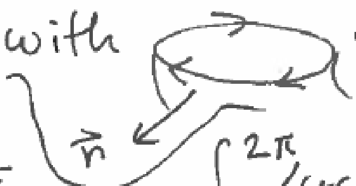
negative	zero	positive

(d) For $\mathbf{E} = \langle z,\ x,\ 2\rangle$, the flux $\iint_S \mathbf{E} \cdot \mathbf{n}\, dS$ is:

-5π	-4π	-3π	-2π	$-\pi$	0	π	2π	3π	4π	5π

Scratch Space

b) Use Stokes' with



C oriented clockwise: $\vec{r} = (\sin t, \cos t, 0)$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle \cos t, -\sin t, 0\rangle \cdot \langle \cos t, -\sin t, 0\rangle\, dt$$

$$= \int_0^{2\pi} 1\, dt = 2\pi$$

c) Let D be the disc in the xy-plane bounded by C, R the region with $\partial R = S + D$. Then $\iiint_R \underbrace{\operatorname{div} G}_{=3}\, dV$

$$= \underbrace{\iint_D \vec{G} \cdot d\vec{S}}_{} + \iint_S \vec{G} \cdot dS$$

$= 0$ as $\vec{n} = (0,0,1)$ and $\vec{G} = (x, y, 0)$

d) Similar, but $\operatorname{div} \vec{E} = 0$

$$\Rightarrow \iint_S \vec{E} \cdot \vec{n}\, dA = \iint_D \vec{E} \cdot (0,0,-1)\, dA$$

$$= \iint_D -2\, dA = -2\pi.$$

Question 6

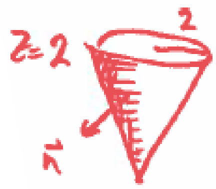
Let $\mathbf{F} = (1 + x + yz)\mathbf{i} + 2y\mathbf{j} + (z + yx)\mathbf{k}$.

(a) Compute curl(**F**). **(1 point)**

$$\begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ 1+x+yz & 2y & z+yx \end{vmatrix} \qquad \langle x-0, -(y-y), 0-z \rangle$$

$$\boxed{\text{curl}(\mathbf{F}) = \langle \; x \;,\; 0 \;,\; -z \; \rangle}$$

(b) Let S be the portion of the cone $z = \sqrt{x^2 + y^2}$ with $z \leq 2$, with outward pointing normal vector. Compute the flux of curl(**F**) through S. **(5 points)**



D= disk
radius 2
~~upward~~ normal
downward

Stokes $\Rightarrow \iint_S \text{curl}\vec{F}\cdot\vec{n}\, dA$

$$= \iint_D \text{curl}\vec{F}\cdot\vec{n}\, dA$$

$$= \iint_D \langle x,0,-z\rangle\langle 0,0,-1\rangle dA$$

$$= \iint_D z\, dA = \iint_D 2\, dA = 2(\pi(2)^2)$$

$$\boxed{\text{flux} = 8\pi}$$

(c) Let E be the sphere $x^2 + y^2 + z^2 = 1$ with outward pointing normal. Compute the flux of **F** through E. **(3 points)**

divergence theorem: B = unit ball

$$\iint_E \vec{F}\cdot\vec{n}\, dA = \iiint_B \text{div}\vec{F}\, dV = \iiint_B (1+2+1)\, dV = 4\cdot\left(\tfrac{4}{3}\pi\right)$$

$$\boxed{\text{flux} = 16/3\,\pi}$$

(d) Is **F** conservative? | Yes | No | **(1 point)**